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REMARKS ON THE STABILITY OF PLANETARY SYSTEMS.

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As, in some quarters, quite erroneous views seem to be entertained regarding the conditions necessary for the stability of the solar system, it may be of service to note here, in brief, what is known on this subject.

It is remarkable that, although the meaning of stability in statics is well known, no one, so far as I know, has ever given a rigorous definition of this term as used in dynamics. As applied to the solar system, the sense attributed to it in general seems to involve the idea that the mean distances, eccentricities and mutual inclinations of the planets should always be comprised within narrow limits. But if this be the proper meaning of the word, one is tempted to ask—how narrow? It is plain, when we consider the matter more closely, that the distinction between stability and instability is one of kind and not of degree. There must be a sharp line separating stable systems from unstable.

In the first place we must discriminate between two possible significations of the term; a system may be stable or unstable with reference to the action of foreign forces, or with reference to the mutual action of its parts. A slight disturbance from without may cause in a moving system only trifling deviations from the previous paths of motion, or the effect may be a greater and still greater departure from them. This is quite analogous to the stability and instability of statics. But the stability of a planetary system, with reference to its own action, must be defined in a way quite peculiar.

A planetary system is stable when finite superior and inferior limits can be assigned to all the distances of the bodies composing it, and that, no matter how long the motion may be prolonged; but if to some or all the distances, no superior limit other than infinity, or no inferior limit other than zero, can be assigned, the system is unstable.

Hence in a stable system, there can be no collision and no indefinite separation of the bodies composing it. With this definition of stability we see that, in the problem of two bodies, motion in an ellipse is stable, but motion in a right line, or parabola, or hyperbola is unstable.

With regard to stable systems, we may enunciate the following proposition: The coordinates of all the bodies in a stable system, or any functions of them which remain always finite and continuous, can be developed in infinite converging series of periodic terms, each of the form $K^{\sin}_{\cos}(k\,t + \beta)$; K, k and β being absolutely constant; and the argument $k\,t + \beta$ is always composed, as a linear function with positive or negative integral coefficients, of other arguments, whose number never exceeds $3n - 3$, n being the number of bodies in the system.

With regard to the convergence of these series, it must be understood that it is asserted only when K and k are taken as wholes; if these quantities are expressed in infinite series involving the powers and products of certain parameters, these series may cease to be convergent long before the system passes from stability to instability. But the convergence of these is altogether another question.

If mathematicians had succeeded in completely integrating in finite terms the differential equations of motion of a system of material points acting on each other in accordance with the law of gravitation, the conditions under which the motion is stable or unstable could be immediately assigned. But there is scarcely any reason to expect that this will ever be accomplished, and perhaps it is in the power of analysis, at present even, to demonstrate its imposibility. For, just as, from the fact that the equation, $\sin x = 0$, has an infinite number of roots, it may be confidently asserted that $\sin x$ cannot be represented in finite terms by algebraic functions; or that, because an elliptic function possesses the property of double periodicity, it cannot be equivalent to any finite expression involving circular or logarithmic functions; so it is probable that the functions, defined by the differential equations of planetary motion, have properties that cannot belong to any finite expression involving quadratures. If this should be the case, all attempts to arrive at a complete solution, in finite terms, of the famous problem of three bodies, must prove as abortive as those made to square the circle or to express elliptic integrals in circular and logarithmic functions.

The solution of the general problem, given the initial positions and velocities of a system of material points, to determine whether the ensuing motion is stable or unstable, in the sense we have attributed to these words, does not seem to have engaged the attention of geometers. It does not however demand the complete integration of the differential equations of motion. Thus where the system contains two material points only two integrals are needed, that of the conservation of living forces, and that of the conservation of areas. And the criterion of stability is here exceedingly simple; it depends on the values of the constants annexed to complete these integrals; to insure stability the first must be negative and the second must not vanish.

The question of stability is intimately connected with the values of the coefficients k in the periodic terms of the series into which the coordinates can be developed in the case of stable motion. If we substitute for the coordinates in the differential equations indeterminate series of this form, we shall arrive at a number of equations exactly sufficient to determine all the coefficients K and the $3\,n — 3$ independent k's in terms of $3n—3$ arbitrary constants which must be given by the initial state of the system. Eliminating all the coefficients K, we shall have left $3\,n — 3$ equations determinining the $3\,n — 3$ independent k's. It can be readily shown that in these equations k appears only in even powers. Then if the initial positions and velocities are such that they make these equations afford positive and finite values for all the quantities k^2, and if it be granted that these are the proper roots to take, motion of the system is undoubtedly stable. But if these equations afford negative values for some or all of these quantities, and it be granted that these are the proper roots to take, the system is necessarily unstable. In the case where some of the quantities k^2 vanish, the question of stability must be determined from other considerations. Whether in any case it is proper to take the imaginary roots of these equations for the quantities k^2, or whether, like some analogous equations in the theory of heat, they have no roots of this kind, is a point which is not yet clear. In all this, as the equations determining the k's can only be obtained in the form of infinite series, it must be shown beforehand that the constants, defining the initial positions and velocities of the bodies of the system, enter into these equations in such a way that they render the series convergent; else any conclusions, as to the values of the k's, deduced from them, are not legitimately established.

The number of conditions, necessary to insure stability, of course increases with the number of bodies composing the system. In all cases the constant annexed to the integral of living forces in relative motion must be negative. It is needless to say that in the solar system this condition is fulfilled. Certain popular writers have got it that incommensurability of mean motions is a *sine qua non* of stability; but I am not aware that this has been asserted by any geometer or astronomer of note.

This mistake doubtless arose from noticing that the near approach of mean motions to commensurability produces inequalities having large coefficients through the division by the small mean motion of the argument. But it will not do to assume that these coefficients increase beyond every limit when the mean motion of the argument diminishes without limit, or that when this vanishes, there are terms in the planetary elements proportional to the time.

Let us illustrate this point more at length. If a is the mean distance of one of the planets, and θ an argument whose mean motion nearly or altogether vanishes; then so far as this argument is concerned we may have the equation

$$\frac{d\,a}{d\,t} = A \sin \theta.$$

If the mean motion of θ is supposed to vanish, the integral of this equation is often written, $a = a_0 + A \sin \theta \,.\, t$, a_0 being the value of a at the origin of time. But although this treatment is allowable when we wish to find the value of a for small values of t, it will not answer when the object is to discover whether a is a periodic function of t or not. For it has been assumed that A is constant, whereas it is a function, not only of a, but of all the other varying elements which define the dimensions of the orbits; also θ is not constant, although its mean motion vanishes, for its periodic inequalities may have some effect here. If the approximation were carried further, it would be found that there were terms in a multiplied by t^2, t^3, &c., and that thus it would be more exact to write

$$a = a_0 + A \sin \theta \,.\, t + B\,t^2 + C\,t^3 + \ldots\ldots\ldots\ldots\ldots$$

What if the right member of this equation should turn out to be the development of a periodic function of t? This in fact is the result in a large class of cases. Thus it is plain that the equation,

$$\frac{d\,a}{d\,t} = A \sin \theta,$$

must be treated as a differential equation; that is its right member must be regarded as an unknown function of t as well as its left.

But several instances in the solar system of commensurability of mean motions, without resultant instability, ought to have prevented this mistake. The three inner satellites of Jupiter have mean motions generally granted to be exactly commensurable, yet the system is not supposed to be unstable. Again what prevents our regarding the moon as a planet revolving around the sun, and our attributing its being sometimes in advance sometimes behind the earth, its having a radius vector, sometimes greater sometimes less than that of the earth to the perturbing influence of the latter? In this view the period of revolution of the moon about the sun is precisely equal to that of the earth; yet there is no instability here. If the planet Venus were moved outward from the sun, until its mean distance from this body became nearly equal to that of the earth, and if at the same time their eccentricities and longitudes of perihelia were so nearly equal as to permit their being for some time in the vicinity of each other, the effect of their mutual action would be to make the mean values of these elements rigorously equal, and each planet would become a satellite to the other. Instability would not result from this disposition.

There are moreover two remarkable particular solutions of the problem of three bodies, in both of which the periods of revolution of the two planets are exactly equal, without instability ensuing. These solutions have been developed by Laplace (Mécanique Céleste, Book X, Chap. VI)· The first, in its stable form, may be stated thus:—Two planets may move in the same direction about the sun in two equal ellipses lying in the same plane, having their foci at the center of the sun, and their greater axes inclined at an angle of 60°, provided the planets are at the same time in corresponding points of their orbits, so that they together with the sun are always at the vertices of an equilateral triangle. The laws of motion are the same as in the case of two bodies, but the common mean motion is given by the formula,

$$n = \sqrt{\frac{m + m' + m''}{a^3}},$$

where m, m', m'' are the three masses, and a the common mean distance.

The second is stated thus:—Two planets may revolve in the same direction about the sun, in similar ellipses having their foci at the center of this body, and their greater axes coincident in direction, provided that the ratio of the axes is determined by a root of a certain equation of the fifth degree, involving in its coefficients the ratios of the three masses. The planets must be at corresponding points in their orbits at the same time, so that they and the sun always lie in a right line. The laws of motion are the same as for two bodies, but the common mean motion is given by a complex expression involving the root of the equation just mentioned.

It may be noticed that Lionville has shown that the latter of these solutions is unstable in the sense we first attributed to this word, that is with reference to slight disturbances from without.* It is probable that the first solution is in the same case, but I do not know that it has ever been discussed in this respect.

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*See *Conn. des Temps* for 1845, or Lionville·s Journal, First Series, Vol. VII, p. 110.

Let us now establish in a clearer light the fact that the commensurability of mean motions does not necessarily produce instability. The solution of the problem of three bodies can be reduced to the integration of a system of eight differential equations of the first order; and by a suitable selection of variables, these may be made to take the canonical shape; that is the differential coefficient of each varying element, with respect to the time, will be equal to the positive or the negative of the partial differential coefficient, with respect to the conjugate element, of a function R, analogous to, but not identical with, the disturbing function in perturbations. Thus the eight elements are divided into two classes; four being functions of the mean distances and eccentricities, relate to the dimensions of the two orbits; while the other four, their conjugates, are simply the elementary arguments of the periodic terms contained by R in its developed form. The selection of these last may be made arbitrarily. If we take one of them, as θ, to coincide with an argument of R, whose mean motion nearly or exactly vanishes, and call the element, conjugate to this, Θ, we shall have the two differential equations

$$\frac{d\,\Theta}{d\,t} = \frac{d\,R}{d\,\theta}, \qquad \frac{d\,\theta}{d\,t} = -\frac{d\,R}{d\,\Theta}.$$

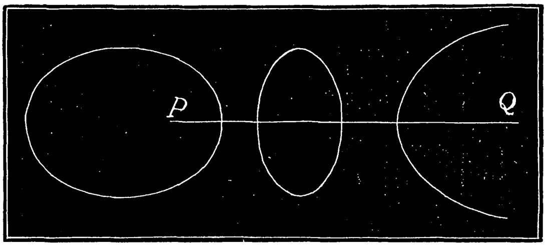
Let us now suppose that R is reduced to its terms which have only θ in their arguments; then

$$R = -B - A \cos\theta - A' \cos 2\theta - A'' \cos 3\,\theta - \ldots\ldots\ldots$$

where B, A, &c., are functions of Θ and the three other elements of its class. As R thus limited does not contain the three elements which are in the class of θ, its partial differential coefficients, with respect to these quantities, vanish. Then the three elements accompaning Θ in its class are constant, and R, as we have limited it, contains no other variables than Θ and θ. Thus if the differential equations determining Θ and θ are multiplied, the first by $d\,\theta$, and the second by $-d\,\Theta$, and the results added, we have an exact differential, which being integrated gives $R =$ a constant, or as it may be written,

$$C = B + A \cos\theta + A' \cos 2\,\theta + A'' \cos 3\,\theta + \ldots\ldots\ldots$$

In order to obtain the values of Θ and θ in terms of t, we should have to make another integration, but this integral suffices to show whether the element Θ, on which depend the dimensions of the orbits, is confined between finite limits. The value of the constant C is readily obtained by substituting in the right member of the equation the values of θ, Θ and the three other elements of its class, which have place at a determinate time, as for instance the epoch from which t is counted. Then this equation may be regarded as the polar equation of a curve, upon which the values of Θ and θ are always found together. Let us suppose that, Θ being taken as the radius and θ as the angle, the equation is represented by a curve having one or more of such branches as those in the figure. P is the pole from which Θ is measured, and PQ the line from which θ is measured. Now if the values of Θ and θ, at a determinate time, are found on the closed branch which envelops the pole P, it is plain Θ will always be comprised between certain finite limits. And in this case the mean motion of the argument θ cannot vanish, as θ moves through the entire circumference. Here it is always possible to develope Θ ahd θ in converging infinite series consisting of periodic terms, such as



$$\Theta = \Theta_0 + \Theta_1 \cos [\theta_0 (t + c)] + \Theta_2 \cos 2 [\theta_0 (t + c)] + \ldots..$$
$$\theta = \theta_0 (t+c) + \theta_1 \sin [\theta_0 (t+c)] + \theta_2 \sin 2 [\theta_0 (t+c)] + \ldots.$$

where Θ_0, Θ_1, Θ_2, &c., θ_0, θ_1, θ_2, &c, and c are constants. These are the series of which Delaunay has made such constant use in his Theory of the Motion of the Moon.

But if the values of Θ and θ at a determinate time, are found upon the closed branch holding the middle place in the figure, Θ will always be contained within finite limits, while θ, its mean motion vanishing, will make oscillations forth and back between definite limits. Hence although the mean motions are here exactly commensurable, no instability results. This case obtains in the three inner satellites of Jupiter, and it also has place in the system of the sun, earth and moon, when the last, as well as the earth, is regarded as a planet circulating about the sun.

In all this we must remember that the values of Θ and the three other elements of its class, both at the origin of time and ever before and after, must be such that they allow the development of R in periodic series to be convergent; else any conclusions derived from this series are not legitimately established.

From this we see that the commensurability or incommensurability of mean motions has no marked connection with stability. This last may be said to depend rather on whether the elements, such as the mean distances and eccentricities, which determine the dimensions of the orbits, have at a given time such values as make the orbits decidedly elliptic, and permit to them the vibrations caused by the action of the members of the system, without inteference or in other words intersections. There can hardly be a doubt that our solar system as composed of the sun and eight principal planets fulfills these conditions.

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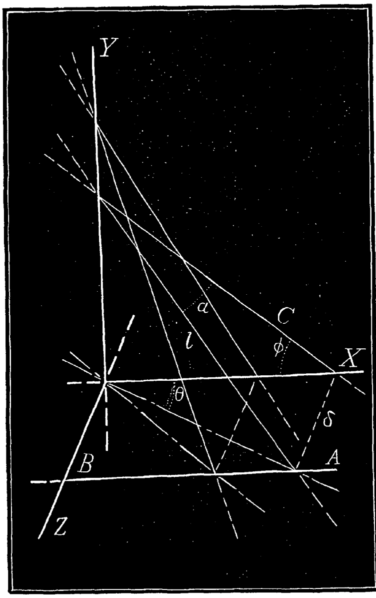
SOLUTION OF A PROBLEM.

BY E. W. HYDE, PROF. OF MATH. IN PA. MILITARY ACADEMY, CHESTER, PA.

Locus of a right line moving so that two fixed points of it remain constantly in two other right lines, which are perpendicular to each other but do not intersect.



Let the axis of y and line $A\ B$ be the directrices.

A line in $X\ Y$ through two points is

$$\frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2}, \quad \ldots\ldots(a)$$

or, $(y-y_1)(x_1-x_2)=(x-x_1)(y_1-y_2)$.

For the line c we have

$$x_1=0,\ y_1=c\sin\phi,$$
$$x_2=c\ c\ s\,\phi,\ y_2=0.$$

$$\text{But }\frac{\delta}{c\cos\phi}=\tan\theta=\frac{z}{x}.$$

$$\therefore\ \cos\phi=\frac{\delta x}{cz}.$$

$$\therefore\ \sin\phi=\sqrt{1-\cos^2\phi}=\sqrt{1-\frac{\delta^2x^2}{c^2z^2}}$$

Substituting these values in equation (a)

$$\left(y-x\sqrt{1-\frac{\delta^2\ {}^2}{c^2z^2}}\right)\left(-\frac{\delta x}{z}\right)=x\left(c\sqrt{1-\frac{\delta^2x^2}{c^2z^2}}\right),$$

$$\text{or } c\,x\left(\frac{\delta}{z}-1\right)\sqrt{1-\frac{\delta^2x^2}{c^2z^2}}=\frac{\delta\,x\,y}{z},$$

and by squaring and reducing, we obtain,
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